A PROBLEM RELATED TO THE APPROXIMATION OF π BY ARCHIMEDES/HUYGENS

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Dedicated to Herman J. J. te Riele on the occasion of his retirement from the CWI in January 2012

1. The problem and its origin.

It is well known that Archimedes approximated 2π (:= the length of the circumference of a circle having radius $r = 1$) by the lengths of inscribed and circumscribed regular n-gons.

Denoting the length of such an inscribed n-gon by ℓ_n and that of a circumscribed one by L_n we have

$$\ell_n = 2n\sin\frac{2\pi}{2n} \qquad \text{and} \qquad L_n = 2n\tan\frac{2\pi}{2n}. \tag{1}$$

Huygens considered the question: Which of ℓ_n and L_n is the best approximation of 2π and to what extent ?

It should be clear that $\ell_n < 2\pi < L_n$. So, for a suitable $\lambda \in (0,1)$ one should take $2\pi = \lambda\ell_n + (1-\lambda)L_n$.

From this it is easily seen that the best λ would be $\lambda = \frac{1}{\left(1+\frac{2\pi-\ell_n}{L_n-2\pi}\right)}$.

So, one should consider the ratio $\frac{2\pi-\ell_n}{L_n-2\pi}$, or as we actually did

$$\frac{L_n-2\pi}{2\pi-\ell_n} = \frac{\tan\frac{\pi}{n}-\frac{\pi}{n}}{\frac{\pi}{n}-\sin\frac{\pi}{n}}.$$

Writing $x := \frac{\pi}{n}$ we are thus led to consider the (even) function $Q(x) := \frac{\tan x - x}{x-\sin x}$ for x close to 0.

It was known to Huygens that $Q(x) > 2$, and using l'Hôpital's rule it is easily seen that $\lim_{x\to 0} Q(x) = 2$.

Consequently one should (in this context) approximate 2π by $\frac{2}{3}\ell_n + \frac{1}{3}L_n$. Also note that $\frac{2}{3}\ell_n + \frac{1}{3}L_n > 2\pi$.

(A similar analysis holds for the areas a_n and A_n of the n-gons.)

For us it was just a matter of curiosity to have a closer look at the coefficients in the power series of the function $\frac{\tan x - x}{x-\sin x}$ for x close to 0.

Invoking *Mathematica* we found (for various values of nMax) for example:

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nMax = 30; (*  For example  *)
Normal[Series[(Tan[x] - x)/(x - Sin[x]), {x, 0, nMax}]]
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$$2+\frac{9\,x^{2}}{10}+\frac{513\,x^{4}}{1400}+\frac{297\,x^{6}}{2000}+\frac{2\,595\,081\,x^{8}}{43\,120\,000}+\frac{136\,726\,449\,x^{10}}{5\,605\,600\,000}+\frac{7\,757\,835\,963\,x^{12}}{784\,784\,000\,000}+$$
$$\frac{4\,810\,522\,436\,537\,x^{14}}{1\,200\,719\,520\,000\,000}+\frac{228\,184\,846\,967\,215\,909\,x^{16}}{140\,532\,212\,620\,800\,000\,000}+\frac{924\,798\,350\,722\,118\,597\,x^{18}}{1\,405\,322\,126\,208\,000\,000\,000}+$$
$$\frac{423\,613\,976\,567\,459\,270\,644\,897\,x^{20}}{1\,588\,323\,173\,482\,805\,760\,000\,000\,000}+\frac{1\,716\,842\,780\,515\,524\,728\,374\,151\,x^{22}}{15\,883\,231\,734\,828\,057\,600\,000\,000\,000}+$$
$$\frac{126\,064\,430\,908\,322\,638\,705\,746\,667\,x^{24}}{2\,877\,667\,867\,251\,201\,024\,000\,000\,000\,000}+\frac{15\,852\,808\,185\,558\,085\,074\,420\,916\,349\,x^{26}}{892\,881\,980\,908\,641\,884\,160\,000\,000\,000\,000}+$$
$$\frac{6\,162\,379\,696\,360\,573\,178\,218\,943\,175\,357\,313\,x^{28}}{856\,398\,823\,168\,714\,776\,773\,222\,400\,000\,000\,000\,000}+\frac{324\,677\,394\,542\,156\,500\,969\,976\,683\,473\,676\,127\,x^{30}}{111\,331\,847\,011\,932\,920\,980\,518\,912\,000\,000\,000\,000\,000}$$

and (observing that all coefficients turned out to be positive) arrived at the *conjecture* that all coefficients c_n in the power series expansion

$$\frac{\tan x - x}{x - \sin x} = \sum_{n=0}^{\infty} c_n\, x^{2n}$$

are strictly positive indeed.

We thus ran into the problem: If true, how can this be proved?

2. A proof of the conjecture.

$Q(x)$ is a meromorphic function on the complex plane. Its poles are those of $\tan x$ at the points $x = (2n+1)\pi/2$ with n an integer and at the zeros of $x - \sin x$, except $x = 0$, which is a removable singularity of $Q(x)$.

We consider the square $R = [-2\pi, 2\pi]^2$. Inside this square there are only four poles of $Q(x)$: at the points $\pm\frac{\pi}{2}$ and $\pm\frac{3\pi}{2}$. To see this it suffices to show that $x - \sin x$ has only one (triple) zero inside R. This can be proved formally by computing the variation of the argument of $x - \sin x$ when moving along the rim of the rectangle with vertices at $\pm 2\pi \pm iT$ with T a big real number.

We compute the residues

$$\operatorname*{Res}_{x=\pi/2} Q(x) = \frac{2}{2-\pi}, \qquad \operatorname*{Res}_{x=-\pi/2} Q(x) = -\frac{2}{2-\pi},$$
$$\operatorname*{Res}_{x=3\pi/2} Q(x) = -\frac{2}{2+3\pi}, \qquad \operatorname*{Res}_{x=-3\pi/2} Q(x) = \frac{2}{2+3\pi}.$$

Hence, we may write

$$(2) \quad Q(x) = \frac{8}{\pi(\pi-2)}\frac{1}{1-4x^2/\pi^2} + \frac{8}{3\pi(2+3\pi)}\frac{1}{1-4x^2/9\pi^2} + h(x).$$

where h is analytic on R.

We thus find the following value for c_n

$$(3)\quad c_n = \frac{8}{\pi(\pi-2)}\Big(\frac{2}{\pi}\Big)^{2n} + \frac{8}{3\pi(2+3\pi)}\Big(\frac{2}{3\pi}\Big)^{2n} + d_n,$$

$$\text{with}\quad d_n = \frac{1}{2\pi i}\int_{\partial R}\frac{h(z)}{z^{2n+1}}\,dz.$$

For $x\in\partial R$ we have $|Q(x)-h(x)|\le\frac14$. In fact for $|x|>2\pi$ we have

$$|Q(x)-h(x)| \le \frac{8}{\pi(\pi-2)}\frac{1}{16-1} + \frac{8}{3\pi(2+3\pi)}\frac{1}{16/9-1} = 0.244234\dots$$

Also, for $x\in\partial R$ we will show that $|Q(x)|\le 2$. Since Q is even, we only have to bound $Q(2\pi+iy)$ and $Q(x+2\pi i)$ for $|y|<2\pi$ and $|x|<2\pi$.

First for y real and $|y|<2\pi$ we have

$$Q(2\pi+iy) = \frac{-2\pi + i(\tanh y - y)}{2\pi + i(y-\sinh y)}.$$

Then $|Q(2\pi+iy)|\le 2$ is equivalent to

$$4\pi^2 + (\tanh y - y)^2 < 16\pi^2 + 4(\sinh y - y)^2$$

or

$$\tanh^2 y - 2y\tanh y < 12\pi^2 + 3y^2 + 4\sinh^2 y - 8y\sinh y.$$

So $|Q(2\pi+iy)|\le 2$ follows from the two elementary inequalities: $\tanh^2 y<1$ and $8y\sinh y < 2+3y^2+4\sinh^2 y$.

On the other side of the rectangle, for $-2\pi<x<2\pi$ we have

$$|Q(x+2\pi i)| = \Big|\frac{\tan(x+2\pi i) - x - 2\pi i}{x+2\pi i-\sin(x+2\pi i)}\Big| \le \frac{\coth 2\pi + |x+2\pi i|}{\sinh 2\pi - |x+2\pi i|}$$

$$\le \frac{\coth 2\pi + 2\sqrt{2}\pi}{\sinh 2\pi - 2\sqrt{2}\pi} = 0.0381898\dots$$

It follows that on ∂R we have $|h(x)|\le|Q(x)-h(x)|+|Q(x)|\le 3$, so that

$$|d_n| \le \frac{1}{2\pi}\int_{\partial R}\frac{3}{(2\pi)^{2n+1}}|dz| \le 24(2\pi)^{-2n-1}.$$

Hence with $|\theta|\le 1$

$$(4)\qquad c_n = \frac{8}{\pi(\pi-2)}\Big(\frac{2}{\pi}\Big)^{2n} + \frac{8}{3\pi(2+3\pi)}\Big(\frac{2}{3\pi}\Big)^{2n} + \theta\cdot 24(2\pi)^{-2n-1}.$$

Since $8/(\pi(\pi-2))$ is about $2.23064\dots$ we have

$$c_n > 2\Big(\frac{2}{\pi}\Big)^{2n} - 24\Big(\frac{1}{2\pi}\Big)^{2n+1} > 0 \qquad \text{for all } n\ge 1$$

completing our proof.

3. Further observations.

In the previous Section we proved that in the power series expansion

$$\frac{\tan x - x}{x - \sin x} = \sum_{n=0}^{\infty} c_n \, x^{2n}$$

all c_n are positive.

Writing $\tan x = \sum_{n=1}^{\infty} t_n \, x^{2n-1}$ and $\sin x = \sum_{n=1}^{\infty} s_n \, x^{2n-1}$ we defined

$$T := \sum_{n=1}^{N} t_n \, x^{2n-1} \quad \text{and} \quad S := \sum_{n=1}^{N} s_n \, x^{2n-1}$$

and observed (using Mathematica) the following :
The coefficients q_n in the power series expansion

$$\frac{\tan x - T}{S - \sin x} = \sum_{n=0}^{\infty} q_n \, x^{2n}$$

(1) are all positive if $N \equiv 1 \pmod 2$
(2) are all negative if $N \equiv 0 \pmod 2$.

We have no proof for this and leave a proof (or refutation) as a challenge to the interested reader. One may want to try things out by means of the following program.

```
In[1]:=
n = 3; (*  Also try some other  n ∈ N  *)
T = Normal[Series[Tan[x], {x, 0, 2 n - 1}]];
S = Normal[Series[Sin[x], {x, 0, 2 n - 1}]];
Print["f = ", f = (Tan[x] - T)/(S - Sin[x])];
nTerms = 24;  (*  For example  *)
Normal[Series[f, {x, 0, nTerms}]]
```

$$f = \frac{-x - \frac{x^3}{3} - \frac{2x^5}{15} + \mathrm{Tan}[x]}{x - \frac{x^3}{6} + \frac{x^5}{120} - \mathrm{Sin}[x]}$$

Out[4]=
$$272 + 114\,x^2 + \frac{6101\,x^4}{132} + \frac{890\,149\,x^6}{47\,520} + \frac{26\,000\,961\,209\,x^8}{3\,424\,861\,440} + \frac{64\,491\,289\,360\,457\,x^{10}}{20\,960\,152\,012\,800} + \frac{30\,254\,970\,559\,608\,601\,x^{12}}{24\,262\,182\,114\,508\,800} + \frac{208\,883\,539\,141\,611\,618\,143\,x^{14}}{413\,311\,124\,757\,080\,309\,760} + \frac{7\,710\,587\,768\,733\,558\,650\,509\,987\,x^{16}}{37\,644\,377\,242\,874\,874\,612\,940\,800} + \frac{28\,124\,851\,654\,909\,083\,303\,025\,556\,651\,x^{18}}{338\,799\,395\,185\,873\,871\,516\,467\,200\,000} + \frac{1\,995\,115\,035\,944\,689\,724\,814\,158\,752\,505\,297\,x^{20}}{59\,300\,735\,738\,173\,875\,479\,270\,286\,950\,400\,000} + \frac{59\,091\,732\,921\,225\,317\,488\,043\,271\,690\,096\,506\,747\,x^{22}}{4\,333\,697\,767\,745\,746\,820\,025\,072\,570\,335\,232\,000\,000} + \frac{3\,507\,664\,213\,216\,293\,552\,099\,055\,375\,264\,386\,853\,121\,x^{24}}{634\,730\,927\,560\,495\,429\,381\,430\,471\,959\,353\,753\,600\,000}$$

A similar analysis of the inner and outer areas a_n and A_n leads to "similar" observations.

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